The discrete KP and KdV equations over finite fields

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Abstract

We propose the algebro-geometric method of construction of solutions of the discrete KP equation over a finite field. We also perform the corresponding reduction to the finite field version of discrete KdV equation. We write down formulas which allow to construct multisoliton solutions of the equations starting from vacuum wave functions on arbitrary non-singular curve.

1 Introduction

The goal of this paper is to present a general algebro-geometric method of construction of solutions to the cellular automaton associated with the discrete Kadomtsev–Petviashvilii (KP) equation [4]

$$\tau(n_1+1,n_2,n_3)\,\tau(n_1,n_2+1,n_3+1)-\tau(n_1,n_2+1,n_3)\,\tau(n_1+1,n_2,n_3+1)+ \\ +\,\tau(n_1,n_2,n_3+1)\,\tau(n_1+1,n_2+1,n_3)=0,$$

and with its reduction to the discrete Korteweg–de Vries (KdV) equation [3]

$$\tau(n_1+1,n_3)\,\tau(n_1,n_3)-\tau(n_1,n_3-1)\,\tau(n_1+1,n_3+1)+ \\ +\,\tau(n_1,n_3+1)\,\tau(n_1+1,n_3-1)=0.$$

It turns out that main algebro-geometric ideas of construction of solutions of the equations can be transferred from the level of Riemann surfaces to the level of algebraic curves over finite fields. Our motivation to extend validity of the discrete equations to finite field domain was presented in the recent paper [2], where we also send for relevant literature.

The layout of the paper is as follows. In Section 2 we present the general algebro-geometric scheme for construction of solutions of the discrete KP equation. In Section 3 we describe the algebro-geometric reduction scheme from KP to the discrete KdV equation. Section 4 is devoted to construction of multisoliton solutions (on nontrivial background) starting from the vacuum wave functions on algebraic curves over finite fields. As a simple application of the general method we construct multisolitonic solutions starting from the algebraic curve of genus zero (the projective line).

2 Solutions of the discrete KP equation from algebraic curves over finite fields

This Section is motivated by algebro-geometric (over the complex field) approach to the discrete KP (or Hirota) equation (see for example [5] and [6]) and by [2], where an equivalent version of the discrete KP equation (the discrete analogue of the Toda field system) was studied in detail in the context of finite field valued solutions.

Consider an algebraic projective curve $\mathcal{C}$, absolutely irreducible, nonsingular, of genus g, defined over the finite field $\mathbb{K} = \mathbb{F}_q$ with q elements, where q is a power of a prime integer p (see, for example [8]). By $\mathcal{C}_{\mathbb{K}}$ we denote the set of $\mathbb{K}$-rational points of the curve. By $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$, i.e., $\overline{\mathbb{K}} = \bigcup_{\ell=1}^{\infty} \mathbb{F}_{q^\ell}$, and by $\mathcal{C}_{\overline{\mathbb{K}}}$ denote the corresponding (infinite) set of $\overline{\mathbb{K}}$-rational points of the curve. The action of the Galois group $G(\overline{\mathbb{K}}/\mathbb{K})$ (of automorphisms of $\overline{\mathbb{K}}$ which are identity on $\mathbb{K}$, see [7]) extends naturally to action on $\mathcal{C}_{\overline{\mathbb{K}}}$.

Let us choose:
1. three points $a_i \in \mathcal{C}_{\mathbb{K}}$, $i = 1, 2, 3$,
2. N points $c_\alpha \in \mathcal{C}_{\overline{\mathbb{K}}}$, $\alpha = 1, \ldots, N$, which satisfy the following $\mathbb{K}$-rationality condition

$$\forall \sigma \in G(\overline{\mathbb{K}}/\mathbb{K}), \quad \sigma(c_\alpha) = c_{\alpha'},$$

3. N pairs of points $d_\beta, e_\beta \in \mathcal{C}_{\overline{\mathbb{K}}}$, $\beta = 1, \ldots, N$, which satisfy the following

$\mathbb{K}$-rationality condition

$$\forall \sigma \in G(\overline{\mathbb{K}}/\mathbb{K}) : \quad \sigma(\{d_\beta, e_\beta\}) = \{d_{\beta'}, e_{\beta'}\},$$

4. g points $f_\gamma \in \mathcal{C}_{\overline{\mathbb{K}}}$, $\gamma = 1, \ldots, g$, which satisfy the following $\mathbb{K}$-rationality condition

$$\forall \sigma \in G(\overline{\mathbb{K}}/\mathbb{K}), \quad \sigma(f_\gamma) = f_{\gamma'},$$

5. the normalization point $a_0 \in \mathcal{C}_{\mathbb{K}}$.

As a rule we consider here only the generic case and assume that all the points used in the construction are generic and distinct. In particular, the divisor $D = \sum_{\gamma=1}^{g} f_\gamma$ is non-special. We remark that it is enough the check the $\mathbb{K}$-rationality conditions in any extension field $\mathbb{L} \supset \mathbb{K}$ of rationality of all the points used in the construction.

Definition 1. Fix $\mathbb{K}$-rational local parameters t_i at a_i, $i = 0, 1, 2, 3$. For any integers $n_1, n_2, n_3 \in \mathbb{Z}$ define the wave function $\psi(n_1, n_2, n_3)$ as a rational function with the following properties
1. it has pole of the order at most $n_1 + n_2 + n_3$ at a_0,
2. the first nontrivial coefficient of its expansion in t_0 at a_0 is normalized to one,
3. it has zeros of order at least n_i at a_i for $i = 1, 2, 3$,
4. it has at most simple poles at points c_α, $\alpha = 1, \ldots, N$,
5. it has at most simple poles at points f_γ, $\gamma = 1, \ldots, g$,
6. it satisfies N constraints

$$\psi(n_1, n_2, n_3)(d_\beta) = \psi(n_1, n_2, n_3)(e_\beta), \quad \beta = 1, \ldots, N.$$

The function $\psi(n_1, n_2, n_3)$ is $\mathbb{K}$-rational, which follows from $\mathbb{K}$-rationality conditions of sets of points in their definition. As usual, zero (pole) of a negative order means pole (zero) of the corresponding positive order. Correspondingly one should exchange the expressions "at most" and "at least" in front of the orders of poles and zeros. By the standard application of the Riemann–Roch theorem (and the genericity assumption) we conclude that the wave function $\psi(n_1, n_2, n_3)$ exists and is unique.

In the generic case, which we assume in the sequel, when the order of the pole of ψ at a_0 is $(n_1 + n_2 + n_3)$ denote by $\zeta_k^{(0)}(n_1, n_2, n_3)$ and $\zeta_k^{(i)}(n_1, n_2, n_3)$, $i = 1, 2, 3$, $\mathbb{K}$-rational coefficients of expansion of ψ at a_0 and at a_i, respectively, i.e.,

$$\psi = \frac{1}{t_0^{(n_1+n_2+n_3)}} \left(1 + \sum_{k=1}^{\infty} \zeta_k^{(0)} t_0^k \right), \qquad \psi = t_i^{n_i} \sum_{k=0}^{\infty} \zeta_k^{(i)} t_i^k .$$

Denote by T_i the operator of translation in the variable n_i, $i = 1, 2, 3$, for example $T_1\psi(n_1, n_2, n_3) = \psi(n_1 + 1, n_2, n_3)$. Uniqueness of the wave function implies the following statement.

Proposition 1. *The function ψ satisfies equations*

$$T_i\psi - T_j\psi + \frac{T_j\zeta_0^{(i)}}{\zeta_0^{(i)}}\psi = 0, \quad i \neq j. \tag{1}$$

Notice that equation (1) gives

$$\frac{T_j\zeta_0^{(i)}}{\zeta_0^{(i)}} = -\frac{T_i\zeta_0^{(j)}}{\zeta_0^{(j)}}, \quad i \neq j. \tag{2}$$

Define $\rho_i = (-1)^{\sum_{j<i} n_j}\zeta_0^{(i)}$, then equation (2) implies existence of a $\mathbb{K}$-valued potential (the τ-function) defined (up to a multiplicative constant) by formulas

$$\frac{T_i\tau}{\tau} = \rho_i, \quad i = 1, 2, 3. \tag{3}$$

Finally, equations (1) give rise to condition

$$\frac{T_2\rho_1}{\rho_1} - \frac{T_3\rho_1}{\rho_1} + \frac{T_3\rho_2}{\rho_2} = 0,$$

which written in terms of the τ-function gives the discrete KP equation [4]

$$(T_1\tau)\,(T_2T_3\tau) - (T_2\tau)\,(T_3T_1\tau) + (T_3\tau)\,(T_1T_2\tau) = 0. \tag{4}$$

Corollary 2. *Notice that multiplication of ρ_i, $i = 1, 2, 3$, by a function of the single argument n_i, do not affects nor existence of the τ function nor equation (4) satisfied by the function.*

3 Reduction to the discrete KdV equation

The discrete KdV equation [3, 6]

$$(T_1\tau)\,\tau - (T_3^{-1}\tau)\,(T_3T_1\tau) + (T_3\tau)\,(T_1T_3^{-1}\tau) = 0, \tag{5}$$

is obtained from the discrete KP equation by imposing constraint

$$T_2T_3\tau = \gamma\tau, \tag{6}$$

where γ is a non-zero constant. Transition of the reduction (6) to the level of the wave function ψ will give the algebro-geometric procedure of construction of solutions of the discrete KdV equation.

Lemma 3. *Assume that on the algebraic curve $\mathcal{C}$ there exists a meromorphic function h with the following properties*
1. it has two simple zeroes at points a_2 and a_3 and no other zeroes,
2. it has double pole at a_0,
3. it satisfies N constraints $h(d_\beta) = h(e_\beta)$, $\beta = 1, \ldots, N$,
4. the first nontrivial coefficient of its expansion in the parameter t_0 at a_0 is normalized to one.
Then the wave function ψ satisfies the following condition

$$T_2 T_3 \psi = h\psi. \tag{7}$$

Remark. Existence of such a function h implies that the algebraic curve $\mathcal{C}$ is hyperelliptic.

Proposition 4. *Let h be the function as in Lemma 4. Assume additionally that*

$$h(a_1) = 1. \tag{8}$$

Denote by δ_2 and δ_3 the first coefficients of local expansion of h in parameters t_2 and t_3 at a_2 and a_3, correspondingly

$$h = t_2(\delta_2 + \ldots), \qquad h = t_3(\delta_3 + \ldots).$$

Then the function

$$\tilde{\tau} = \tau\, \delta_2^{-n_2(n_2-1)/2} (-\delta_3)^{-n_3(n_3-1)/2} \tag{9}$$

satisfies the discrete KdV equation (5).

Proof. Expanding equation (7) at a_1 and using of the additional assumption (8) we obtain that

$$T_2 T_3 \rho_1 = \rho_1.$$

Expansions of equation (7) at a_2 and a_3 give

$$T_2 T_3 \rho_2 = \delta_2 \rho_2, \qquad T_2 T_3 \rho_3 = -\delta_3 \rho_3.$$

Therefore the functions

$$\tilde{\rho}_1 = \rho_1, \qquad \tilde{\rho}_2 = \delta_2^{-n_2}\rho_2, \qquad \tilde{\rho}_3 = (-\delta_3)^{-n_3}\rho_3,$$

satisfy condition

$$T_2 T_3 \tilde{\rho}_i = \tilde{\rho}_i, \qquad i = 1,2,3. \tag{10}$$

By Corollary 2 the functions $\tilde{\rho}_i$, $i = 1,2,3$, define new potential $\tilde{\tau}$, connected with τ by (9), which satisfies the discrete KP equation (4). Moreover, conditions (10) imply that $\tilde{\tau}$ is subjected to constraint (6). □

Remark. Notice that in the above procedure one obtains a family, labelled by the parameter n_2, of solutions of the discrete KdV equation.

4 Construction of solutions of the discrete KP and KdV equations using vacuum functions

In this Section we write down results which allow to construct the N-soliton τ-function starting from vacuum ($N = 0$) wave functions on algebraic curve over a finite field. The methods to obtain these results are the same as in the corresponding section of [2].

In the case $N = 0$ let us add superscript 0 to all functions defined above. Define auxiliary vacuum wave functions ϕ^0_α, $\alpha = 1, \dots, N$, as follows.

Definition 2. Fix local parameters t_α at c_α, $\alpha = 1, \dots, N$. For any α define the function ϕ^0_α by the following set of conditions:
1. it has pole of the order at most $n_1 + n_2 + n_3 - 1$ at a_0,
2. it has zeros of order at least n_i at a_i, for $i = 1,2,3$,
3. it has at most simple pole at the point c_α,
4. the first nontrivial coefficient of its expansion in t_α at c_α is normalized to one,
5. it has at most simple poles at points f_γ, $\gamma = 1, \dots, g$.

Using the Riemann-Roch theorem it can be shown that the function ϕ^0_α exists and is unique.

Proposition 5. *Denote by $\psi^0(\boldsymbol{d}, \boldsymbol{e})$, the column with N entries of the form*

$$\left[\psi^0(\boldsymbol{d}, \boldsymbol{e})\right]_\beta = \psi^0(d_\beta) - \psi^0(e_\beta), \quad \beta = 1, \dots, N,$$

denote by $\phi^0_{\boldsymbol{A}}$ *the row with* N *entries*

$$\left[\phi^0_{\boldsymbol{A}}\right]_\alpha = \phi^0_\alpha, \quad \alpha = 1,\dots,N,$$

and denote by $\phi^0_{\boldsymbol{A},m}(\boldsymbol{d},\boldsymbol{e})$ *the* $N \times N$ *matrix whose element in row* β *and column* α *is*

$$\left[\phi^0_{\boldsymbol{A}}(\boldsymbol{d},\boldsymbol{e})\right]_{\alpha\beta} = \phi^0_\alpha(d_\beta) - \phi^0_\alpha(e_\beta), \quad \alpha,\beta = 1,\dots,N.$$

Then the wave function ψ *of the discrete KP equation reads*

$$\psi = \psi^0 - \phi^0_{\boldsymbol{A}}[\phi^0_{\boldsymbol{A}}(\boldsymbol{d},\boldsymbol{e})]^{-1}\psi^0(\boldsymbol{d},\boldsymbol{e}).$$

In the generic case denote by $H^0_{0,\alpha}$ the first nontrivial coefficient of expansion of the function ϕ^0_α at a_0 in the uniformization parameter t_0,

$$\phi^0_\alpha = \frac{1}{t_0^{(n_1+n_2+n_3-1)}}\left(H^0_{0,\alpha} + \dots\right).$$

Corollary 6. *The corresponding expressions for* ρ_i *read*

$$\rho_i = \rho_i^0\left(1 + (T_i H^0_{0,\boldsymbol{A}})[\phi^0_{\boldsymbol{A}}(\boldsymbol{d},\boldsymbol{e})]^{-1}\psi^0(\boldsymbol{d},\boldsymbol{e})\right), \qquad i = 1,2,3,$$

where $H^0_{0,\boldsymbol{A}}$ *is the row with* N *entries* $H^0_{0,\alpha}$.

Proposition 7. *The* τ*-function can be constructed by the following formula*

$$\tau = \tau^0 \det \phi^0_{\boldsymbol{A}}(\boldsymbol{d},\boldsymbol{e}). \tag{11}$$

Corollary 8. *Starting with* $\mathbb{K}$*-valued function* τ^0 *and the local parameters* t_α *at* c_α *chosen in a consistent way with the action of the Galois group* $G(\overline{\mathbb{K}}/\mathbb{K})$ *on* $\mathcal{C}_{\overline{\mathbb{K}}}$ *we obtain* $\mathbb{K}$*-valued function* τ.

Corollary 9. *Notice that equation* (9) *implies that the same formula* (11) *holds also for the* $\tilde{\tau}$*-function in the reduction from the discrete KP equation to the discrete KdV equation.*

We present here explicit formulas for the vacuum functions in the simplest case $g = 0$. In constructing the vacuum functions we will use the standard parameter t on the projective line $\mathbb{P}(\mathbb{K})$ and we put $a_0 = \infty$. Explicit form of the vacuum wave function reads

$$\psi^0 \;\; = \;\; (t-a_1)^{n_1}(t-a_2)^{n_2}(t-a_3)^{n_3}$$

which gives formulas for the functions ρ_1, ρ_2 and ρ_3

$$\begin{aligned}
\rho_1^0 &= (a_1 - a_2)^{n_2}(a_1 - a_3)^{n_3}, \\
\rho_2^0 &= (-1)^{n_1}(a_2 - a_1)^{n_1}(a_2 - a_3)^{n_3}, \\
\rho_3^0 &= (-1)^{n_1+n_2}(a_3 - a_1)^{n_1}(a_3 - a_2)^{n_2}.
\end{aligned}$$

Explicit form of the vacuum τ-function reads

$$\tau^0 = (a_1 - a_2)^{n_1 n_2}(a_1 - a_3)^{n_1 n_3}(a_2 - a_3)^{n_2 n_3}.$$

The auxiliary vacuum wave functions ϕ^0_α, $\alpha = 1, \ldots, N$ have the form

$$\phi^0_\alpha = \frac{1}{t - c_\alpha} \cdot \frac{(t - a_1)^{n_1}(t - a_2)^{n_2}(t - a_3)^{n_3}}{(c_\alpha - a_1)^{n_1}(c_\alpha - a_2)^{n_2}(c_\alpha - a_3)^{n_3}}. \tag{12}$$

In the case of the reduction from KP to KdV the function h reads

$$h(t) = (t - a_2)(t - a_3),$$

while the points a_i, $i = 1, 2, 3$, are subjected to the condition

$$h(a_1) = (a_1 - a_2)(a_1 - a_3) = 1.$$

Then, in notation of Section 3,

$$\delta_2 = a_2 - a_3 = -\delta_3,$$

and, according to Proposition 4, the vacuum solution of the discrete KdV equation reads

$$\tilde{\tau}^0 = (a_1 - a_3)^{n_1(n_3 - n_2)}(a_2 - a_3)^{n_2 - (n_3 - n_2)(n_3 - n_2 - 1)/2}. \tag{13}$$

Finally, by Corollary 9, formulas (11)–(13) allow to find pure N-soliton solutions of the discrete KdV equation over finite fields.

Let us present an example of finite field valued solution of the discrete KdV equation in bilinear form. We take $\mathbb{K} = \mathbb{F}_5$, and all the points used in the construction will be in $\mathbb{F}_{5^2}$, which we consider as extension of $\mathbb{F}_5$ by the polynomial $w(x) = x^2 + x + 1$. The corresponding Galois group reads $G(\mathbb{F}_{5^2}/\mathbb{F}_5) = \{id, \sigma\}$, where σ is the Frobenius automorphism [7]. The parameters of the solution are chosen as follows:

$a_1 = (00)$, $a_2 = (02)$, $a_3 = (03)$,

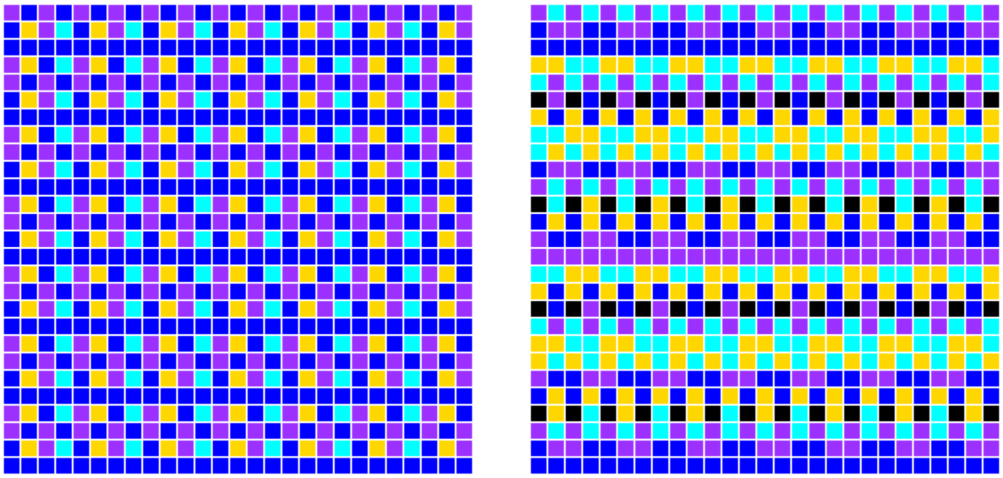

Figure 1: The vacuum and 2-soliton solutions of the discrete KdV equation in $\mathbb{F}_5$; n_1 range from 0 to 26 (directed to the right), n_3 range from 0 to 26 (directed up).

$c_1 = (10)$, $c_2 = \sigma(c_1) = (44)$,
$d_1 = (21)$, $e_1 = \sigma(d_1) = (34)$,
$d_2 = (13)$, $e_2 = \sigma(d_2) = (42)$.

The function $\tilde{\tau}$ is normalized to one for $n_1 = n_2 = n_3 = 0$. This solution of the discrete KdV equation, for $n_2 = 0$, is presented in comparison with the vacuum solution in Figure 1. The elements of $\mathbb{F}_5$ are represented by: – (00), – (01), – (02), – (03), – (04). The periods in variables n_1, n_3 are 4, 24, correspondingly (both must be divisors of $24 = |\mathbb{F}^*_{5^2}|$, see [2]).

Finally, we would like to remark that examples of multisoliton solutions on curves of non-zero genus are more involved and their construction needs some techniques on Jacobians of algebraic curves [1].

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